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On the solution of the coagulation equation with a time-dependent source–application to pulsed injection

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Abstract. A general theoretical treatment is developed for the solution of the time-dependent coagulation equation (with constant coagulation kernel) in the presence of a source term possessing arbitrary time dependence. It is shown how the relevant nonlinear first-order differential equation can be transformed into a linear second-order equation, which can then be used to obtain the general solution of the problem together with information about its asymptotic long-term behaviour. The technique is applied to a periodic source term where it is found that the long-term behaviour of the solution exhibits the same periodicity as the source. Detailed results are derived for particular source terms.

1. Introduction

A considerable amount of effort has been put into the solution of the coagulation equation since the pioneering work of Smoluchowski (1917). However, the majority of this has been devoted to the treatment of an isolated system of coagulating clusters where the total amount of particulate material is constant—see Drake (1972), Twomey (1977), Pruppacher and Klett (1980), Williams and Loyalka (1991) and references contained therein. One of the more important generalizations of this simple picture is the introduction into the coagulation equation of a source term, and it would appear that very little has been done hitherto on a systematic approach to the resulting problem for the situation where this source term possesses arbitrary time variation. As a first step in this direction we proceed by considering, in the present paper, such a systematic approach for the case of a constant coagulation kernel.

In section 2 we introduce a suitable generating function in order to allow the problem to be formulated as a nonlinear differential equation and in section 3 we show how this can be transformed into a linear differential equation, albeit of higher order. We tackle the latter by utilizing standard techniques for such equations, and in section 4 apply our method to a source term which is periodic in time, making use of Floquet's theorem to establish general results concerning the behaviour of the solution. Finally, in section 5 we illustrate our approach with a specific periodic source term.

We note that the assumption of a constant coagulation kernel is known to be a good approximation for Brownian coagulation of an aerosol or hydrosol (Friedlander 1977) and hence our treatment is relevant to the coagulation of such sols with particle sources possessing arbitrary time dependence—in particular, the results of sections 4 and 5 apply to pulsed particle sources. Our approach also provides a benchmark result for the validation of the numerical methods which may well be necessary when other coagulation mechanisms with volume dependent kernels are taken into account.

2. Formulation of the problem

We postulate the coagulation of initially identical particles into clusters, and let $n_r(t)$ be the number density at time t of clusters containing r particles. We suppose that $n_r(0)$ is given and that for $t > 0$ there is a particle source represented by $S_r(t)$. (If no coagulation has occurred prior to $t = 0$ and if the source injects only single particles into the system, then for $r > 1$, $n_r(0) = 0$ and $S_r(t) = 0$.) Now for the constant coagulation kernel P, $n_r(t)$ satisfies the equation.

$$\frac{\mathrm{d}n_r}{\mathrm{d}t} = \frac{1}{2}P\sum_{p=1}^{r-1} n_p n_{r-p} - Pn_r\sum_{p=1}^{\infty} n_p + S_r \qquad (t > 0, r \geqslant 1) \tag{1}$$

together with the initial boundary condition that $n_r(0)$ is given. We non-dimensionalize equation (1) by defining

$$\tau = t/t_0 \qquad m_r = t_0 P\frac{n_r}{2} \qquad W_r = t_0^2 P\frac{S_r}{2} \tag{2}$$

for some constant t_0, when equation (1) takes the form

$$\frac{\mathrm{d}m_r}{\mathrm{d}\tau} = \sum_{p=1}^{r-1} m_p m_{r-p} - 2m_r\sum_{p=1}^{\infty} m_p + W_r \qquad (t > 0, r \geqslant 1) \tag{3a}$$

together with the boundary conditions

$$m_r(0) = q_r \ \text{(given)}. \tag{3b}$$

We begin the solution of equations (3) by defining a generating function $C(\tau, z)$ by

$$C(\tau, z) = \sum_{r=1}^{\infty} m_r z^r \qquad T(\tau, z) = \sum_{r=1}^{\infty} W_r z^r \tag{4}$$

and since $C(\tau, 1)$ and $T(\tau, 1)$ are respectively proportional to the total particle number density and the total particle source term, it is to be expected that the power series expansions (4) are both convergent for $0 \leqslant z \leqslant 1$. It then readily follows from equation (3a) that $C(\tau, z)$ satisfies

$$\frac{\mathrm{d}C(\tau, z)}{\mathrm{d}\tau} = [C(\tau, z)]^2 - 2C(\tau, 1)C(\tau, z) + T(\tau, z) \tag{5a}$$

and therefore

$$\frac{\mathrm{d}C(\tau, 1)}{\mathrm{d}\tau} = -[C(\tau, 1)]^2 + T(\tau, 1). \tag{5b}$$

Letting

$$F(\tau, z) = C(\tau, 1) - C(\tau, z) \qquad V(\tau, z) = T(\tau, 1) - T(\tau, z) \tag{6}$$

now yields

$$\frac{\mathrm{d}F(\tau, z)}{\mathrm{d}\tau} = -[F(\tau, z)]^2 + V(\tau, z) \tag{7a}$$

with boundary condition

$$F(0, z) = \sum_{r=1}^{\infty} q_r(1 - z^r) = F_0(z) \ \text{(given)}. \tag{7b}$$

We note that the total number of clusters at time t, $N(t)$ is given by

$$N(t) = \sum_{r=1}^{\infty} n_r(t) = (2/t_0P)C(\tau, 1) = (2/t_0P)F(\tau, 0) \tag{8}$$

and hence may be immediately obtained from the solution of equation (7a).

Finally, we remark that if there exists additionally a mechanism for the removal of clusters from the system at a constant rate α (for example, by deposition on the walls of the containing vessel) then the original equation (1) becomes modified by the addition of a term $-\alpha n_r$ on the right-hand side. If, for this situation we now define

$$F(\tau, z) = C(\tau, 1) - C(\tau, z) + \beta \qquad V(\tau, z) = T(\tau, 1) - T(\tau, z) + \beta^2 \tag{9}$$

with $\beta = (1/2)\alpha t_0$, it transpires that F satisfies equation (7a).

3. Method of solution

In order to tackle the nonlinear equation (7a) we now define the function $G(\tau, z)$ by

$$G(\tau, z) = \exp\left[\int F(\tau, z)\,\mathrm{d}\tau\right] \tag{10a}$$

and therefore

$$F = \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}\tau}. \tag{10b}$$

It then readily follows from equation (7a) that G satisfies

$$\left(\frac{\mathrm{d}^2G}{\mathrm{d}\tau^2}\right) - VG = 0 \tag{11}$$

which being a linear equation in G is significantly easier to deal with than the nonlinear equation for F. Now since equation (11) is of second-order its general solution will be a linear combination of any two linearly independent solutions, and will thus contain two arbitrary constants. One of these will, however, cancel out when we obtain F from equation (10b), and so if we denote two linearly independent solutions of equation (11) be G_1 and G_2, we deduce from equation (10b) that the general solution for F is given by

$$F(\tau, z) = \frac{G_1'(\tau, z) + \gamma(z)G_2'(\tau, z)}{G_1(\tau, z) + \gamma(z)G_2(\tau, z)} \tag{12}$$

where the dash notation represents differentiation with respect to τ, and $\gamma(z)$ is an arbitrary function of the parameter z. The fact that this general solution for F contains only a single arbitrary quantity γ is, of course, expected since F satisfies the first-order equation (7a). The required unique solution for F is now obtained by application of the boundary condition (7b) to yield $\gamma(z)$. This gives

$$\frac{G_1'(0, z) + \gamma(z)G_2'(0, z)}{G_1(0, z) + \gamma(z)G_2(0, z)} = F_0(z) \tag{13}$$

from which $\gamma(z)$ follows immediately in terms of G_1 and G_2. (In the particular case when $F_0(z) = 0$ corresponding to no particles being initially present, $\gamma(z) = -G_1'(0, z)/G_2'(0, z)$.) Equation (12) in conjunction with equation (8) then yields $N(t)$ for $t > 0$, while expanding $F(\tau, z)$ as a power series in z gives m_r ($r \geqslant 1, t > 0$) as minus the coefficient of z^r.

We now consider the most common situation in which the source injects single particles (as distinct from clusters) into the system. It then follows from equations (4) and (6) that equation (11) becomes (with $W \equiv W_1$)

$$G''(\tau, z) - (1 - z)W(\tau)G(\tau, z) = 0. \tag{14}$$

For certain simple $W(\tau)$ it is possible to solve this analytically to yield a solution for G depending in some explicit fashion on the parameter z. When this is so the above programme may be readily implemented, with $F(\tau, z)$ first being found from equation (12) and hence N and m_r by an expansion of F as a power series in z; we shall give an example of this presently. For more general $W(\tau)$, however, this procedure is not practicable and instead we expand G itself as a power series in z, letting

$$G(\tau, z) = \sum_{r=0}^{\infty} z^r G_{(r)}(\tau). \tag{15}$$

On substituting this into equation (14) we then obtain

$$G''_{(0)} - WG_{(0)} = 0 \tag{16a}$$

$$G''_{(r)} - WG_{(r)} = -WG_{(r-1)} \qquad (r \geqslant 1). \tag{16b}$$

It is clear that equations (16) can now be solves sequentially ($r = 0, 1, 2, \ldots$) by whatever technique (analytical, possibly with a Green's function, or numerical) is appropriate for the given $W(\tau)$. In implementing this procedure we suppose that G_1 and G_2 are each characterized by a specific linear, homogeneous boundary condition and that this condition is applied consistently at each stage of the solution of equations (16). If equations (16) are then solved for $r = 0, 1, 2, \ldots, M$, equations (12) and (13) allow the first $(M + 1)$ terms of the power series for F to be found and hence $N(\tau)$ together with $m_r(\tau)(1 \leqslant r \leqslant M)$ to be obtained.

At this stage of the work it is worth confirming that our general approach through equations (12) and (13) will yield the physically necessary result that $F(\tau, z) \geqslant 0\,(\tau > 0)$ if $F(0, z) \geqslant 0$. To prove this, we multiply equation (11) by $G(\tau, z)$ and integrate with respect to τ from 0 to τ. After a little manipulation this yields the result

$$\frac{G'(\tau)}{G(\tau)} = \frac{1}{[G(\tau)]^2}\left\{[G(0)]^2\left[\frac{G'(0)}{G(0)}\right] + \int_0^{\tau}\{[G'(\tau')]^2 + V(\tau')[G(\tau')]^2\}\,\mathrm{d}\tau'\right\} \tag{17}$$

from which it follows immediately that $[G'(\tau)/G(\tau)] \geqslant 0$ since $[G'(0)/G(0)] \geqslant 0$.

Finally, we point out a basic similarity between the long-term behaviour of the solution F (equation (12)) of our present problem (equation (7)) and the long-term behaviour of the solution E of the corresponding linear equation $\mathrm{d}E(\tau)/\mathrm{d}\tau = -E(\tau) + V(\tau)$. In the latter case the general solution may be expressed as

$$E(\tau) = E_1(\tau) + \beta E_2(\tau)$$

where E_1 and E_2 respectively satisfy the equations $\mathrm{d}E_1/\mathrm{d}\tau = -E_1 + V$ and $\mathrm{d}E_2/\mathrm{d}\tau = -E_2$, and β is an arbitrary constant. A unique solution is given for $E(\tau)$ by specifying $E(0)$ which then determines β. Now if, as is often the case, $\lim_{\tau\to\infty}[E_2(\tau)/E_1(\tau)] = 0$, then the long-term behaviour of $E(\tau)$ becomes independent of β and hence independent of the initial value $E(0)$. The term $\beta E_2(\tau)$ is then described as forming the 'transient component of E' and the long-term behaviour of E becomes independent of this transient component and depends only on the driving term $V(\tau)$. We can readily see that such transient behaviour can also occur with the solution (12) of our nonlinear problem (7). If a pair of linearly independent solutions G_1 and G_2 of equation (11) can be chosen such that $\lim_{\tau\to\infty}[G_2(\tau)/G_1(\tau)] = 0 = \lim_{\tau\to\infty}[G_2'(\tau)/G_1'(\tau)]$, then equation (12) gives $\lim_{\tau\to\infty} F(\tau, z) = G_1'(\tau, z)/G_1(\tau, z)$, which is independent of $\gamma(z)$ and hence independent of $F(0, z)$. Under these circumstances the initial cluster distribution has only a transient effect on its future development and its long-term behaviour is determined entirely by the source term.

The results obtained in this section can be usefully illustrated by the case of a source injecting single particles with time variation $(\tau+\tau_0)^{-2}$. That is, we take $W_1(\tau) = A/(\tau+\tau_0)^2$ (A and τ_0 are constants) and $W_r = 0$ ($r \geqslant 2$). Equation (14) then becomes

$$(\tau + \tau_0)^2 G'' - A(1-z)G = 0 \tag{18}$$

with linearly independent solutions $G_1 = (\tau + \tau_0)^{q_1}$ and $G_2 = (\tau + \tau_0)^{q_2}$ where $q_1 = (1/2)[1 + (1 + 4A(1-z))^{1/2}]$, $q_2 = -(1/2)[(1 + 4A(1-z))^{1/2} - 1]$, and thus the general solution for F is given by

$$F = \frac{q_1(\tau + \tau_0)^{q_1-1} + \gamma(z)q_2(\tau + \tau_0)^{q_2-1}}{(\tau + \tau_0)^{q_1} + \gamma(z)(\tau + \tau_0)^{q_2}}. \tag{19}$$

For given $F_0(z)$, $\gamma(z)$ can then be determined by equation (13) and hence m_r obtained by the expansion of F as a power series in z. Now, it is clear that as $\tau \to \infty$, the second term in both the numerator and the denominator of F can be neglected as compared with the first term and hence, in accordance with the remarks of the previous paragraph we find that the long-term behaviour of F is independent of the initial conditions, being given by

$$F(\tau, z) = \frac{q_1}{(\tau + \tau_0)}. \tag{20}$$

Expressions for N and n_r follow immediately, all being proportional to $(\tau + \tau_0)^{-1}$. Finally, we make the point that the long-term behaviour (20) (though not the general result (19)) will also hold when $W_1(\tau)$ takes the more general form

$$W_1(\tau) = A(\tau + \tau_0)^{-2} + \sum_{p=3}^{\infty} A_p(\tau + \tau_0)^{-p}. \tag{21}$$

Following the approach of Ince (1927), it is seen that equation (14) then has a regular singularity at $\tau = \infty$. Two linearly independent solutions of equation (14) then exist, each of which can be expressed as a power series in $(\tau + \tau_0)^{-1}$ which is convergent for sufficiently large τ. The leading terms in these two power series are the solutions G_1 and G_2 of equation (18) given above, and hence it follows that the long-term behaviour of $F(\tau, z)$ will be determined by equation (20).

4. Periodic source term

We consider now the situation when the source term is periodic in τ with period T and hence in equation (11), $V(\tau + T) = V(\tau)$. Under these circumstances we can utilize Floquet's theorem and its consequences which apply to general linear differential equations whose coefficients are periodic functions of the independent variable. The relevant theory is developed in, for example, Ince (1927), Arscott (1964) and Wilson (1954), and here we shall only briefly recapitulate the essential parts which are relevant to our application of the theory.

Let $\psi_1(\tau)$ and $\psi_2(\tau)$ be any two linearly independent solutions of equation (11). Then since $V(\tau + T) = V(\tau)$, $\psi_1(\tau + T)$ and $\psi_2(\tau + T)$ will both satisfy equation (11) and thus

$$\begin{aligned}\psi_1(\tau+T) &= \alpha_{11}\psi_1(\tau)+\alpha_{21}\psi_2(\tau)\\ \psi_2(\tau+T) &= \alpha_{12}\psi_1(\tau)+\alpha_{22}\psi_2(\tau)\end{aligned} \tag{22}$$

for some constants α_{11}, α_{21}, α_{12} and α_{22}. We are interested in obtaining solutions of equation (11), $\Gamma(\tau)$ which satisfy

$$\Gamma(\tau+T) = \sigma\Gamma(\tau) \tag{23}$$

for some constant σ. Such $\Gamma(\tau)$ must be capable of being expressed in the form

$$\Gamma(\tau) = A_1\psi_1(\tau)+A_2\psi_2(\tau) \tag{24}$$

for constants A_1, A_2 and from equations (22)–(24) we then readily obtain

$$\begin{pmatrix}\alpha_{11}-\sigma & \alpha_{12}\\ \alpha_{21} & \alpha_{22}-\sigma\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix} = 0 \tag{25}$$

showing that σ and $\begin{pmatrix}A_1\\ A_2\end{pmatrix}$ are respectively eigenvalues and eigenvectors of the (2×2) matrix $\boldsymbol{\alpha}$. We therefore expect to obtain two linearly independent Γ satisfying equation (23) and these may be identified with the solutions G_1 and G_2 of equation (11), used in section (3). We now choose ψ_1 and ψ_2 to satisfy the conditions

$$\psi_1(0)=1 \qquad \psi_1'(0)=0 \qquad \psi_2(0)=0 \qquad \psi_2'(0)=1 \tag{26}$$

when it is readily shown from equations (22) that $\alpha_{11}=\psi_1(T)$, $\alpha_{12}=\psi_2(T)$, $\alpha_{21}=\psi_1'(T)$ and $\alpha_{22}=\psi_2'(T)$, and that the quadratic equation for the eigenvalues σ takes the form

$$\sigma^2 - Q\sigma + 1 = 0 \tag{27}$$

where $Q=\psi_1(T)+\psi_2'(T)$. Now $V(\tau,z)>0$, from which it follows (see Ince (1927)) that $Q>2$ and hence that σ is real. We label the solutions of equation (27) as $\sigma_1(>1)$, $\sigma_2(=\sigma_1^{-1}<1)$ and the corresponding solutions of equation (11) are then given by

$$G_p(\tau) = \psi_2(T)\psi_1(\tau)+[\sigma_p-\psi_1(T)]\psi_2(\tau) \qquad (p=1,2). \tag{28}$$

We now suppose τ to lie in the interval $[0,T]$ and consider a dimensionless time $\tau'=nT+\tau$, where n is a positive integer. It then follows from equations (12) and (23) that

$$F(\tau',z) = \frac{G_1'(\tau,z)+\sigma_2^{2n}\gamma(z)G_2'(\tau,z)}{G_1(\tau,z)+\sigma_2^{2n}\gamma(z)G_2(\tau,z)} \tag{29}$$

where $\gamma(z)$ may be found from equation (13). It is clear that for values of τ' for which the second terms in the numerator and the denominator are significant, $F(\tau',z)$ will not be periodic in τ'. However, for sufficiently large values of τ' (corresponding to $n\gg1$) these second terms will become insignificant since $\sigma_2<1$, and hence the long-term behaviour of $F(\tau',z)$ will be that of a periodic function with period T, being given by

$$F(\tau+nT,z) = \frac{G_1'(\tau,z)}{G_1(\tau,z)}. \tag{30}$$

In accordance with the remarks of the previous section, this corresponds physically to the cluster size distribution function becoming periodic (with the source term period) after a time sufficiently long for the initial transient contribution to have become negligible. This time will be significantly greater than $T/(2\ln\sigma_1)$.

5. Applications

On the basis of the last section, the procedure to be followed for a periodic source term with single-particle injection is as follows. Two linearly independent solutions of equation (14) must first be found, from which we can then construct the functions ψ_1 and ψ_2 of section 4. These allow σ_2 to be obtained from equation (27) and hence G_1 and G_2 to be constructed from equations (28). If we are interested in the solution F for general τ' and specified boundary conditions at $\tau' = 0$ we then use equation (29) with $\gamma(z)$ determined from these boundary conditions, while if only the long-term behaviour of F is of interest we may obtain this from equation (30). In implementing this procedure we recall the point made earlier that for certain simple $W(\tau)$ the solutions of equation (14) can be obtained in terms of known functions involving z as a parameter. This then allows $F(\tau', z)$ to be found as a function of z from which $N(\tau')$ and $m_r(\tau')$ may finally be calculated by expanding F as a power series in z. For more complicated $W(\tau)$ it is necessary to obtain linearly independent solutions of equation (14) as explicit power series in z as described after equation (15). If this is done for $G_{(r)}(\tau)$ with $r = 0, 1, \ldots, M$, then implementation of the procedure outlined above will finally allow the calculation of $N(\tau')$ and $m_r(\tau')$ for $r = 1, 2, \ldots, M$.

Now, if we suppose $W(\tau)$ to be not only periodic but also continuous, then even the relatively simple form $W(\tau) = A + B\cos(2\pi\tau/T)$ gives rise to the somewhat complicated Mathieu functions when we look for solutions of equation (14). For a more general form of $W(\tau)$ equation (14) becomes Hill's equation (Arscott 1964) with even greater complexity in its solution. In order to illustrate the theory of the last section, we shall therefore consider a particular discontinuous form for our periodic function $W(\tau)$. Our choice is dictated on the one hand by the requirement that we should be able to carry through the detailed calculations analytically, and on the other by a desire that $W(\tau)$ should correspond to a source which can be generated experimentally. The form we take is $W(\tau)$ with period T defined in the interval $[0, T]$ by

$$W(\tau) = \begin{cases} 0 & (0 \leqslant \tau \leqslant \rho) \\ K^2 & (\rho < \tau < T) \end{cases} \tag{31}$$

with given constants K and $\rho (< T)$.

We begin by solving equation (14) separately in the two intervals $[0, \rho]$ and $[\rho, T]$. Since W and G are always finite, equation (14) implies that G'' is always finite, from which it follows that G' and G are both continuous at $\tau = \rho$. This then allows the general solution of equation (14) in the interval $[0, T]$ to be found. This solution contains two arbitrary constants which can then be determined in order to yield the solutions $\psi_1(\tau)$ and $\psi_2(\tau)$ satisfying the conditions (26). The results obtained are

$$\psi_1(\tau) = \begin{cases} 1 & (0 \leqslant \tau \leqslant \rho) \\ \cosh[K(1-z)^{1/2}(\tau - \rho)] & (\rho \leqslant \tau < T) \end{cases} \tag{32a}$$

$$\psi_2(\tau) = \begin{cases} \tau & (0 \leqslant \tau \leqslant \rho) \\ \rho\cosh[K(1-z)^{1/2}(\tau-\rho)] + \dfrac{\sinh[K(1-z)^{1/2}(\tau-\rho)]}{K(1-z)^{1/2}} & (\rho \leqslant \tau \leqslant T). \end{cases} \tag{32b}$$

We now proceed to calculate σ_1 and σ_2 from equation (27) and obtain

$$\sigma_1 = \cosh\theta + a\sinh\theta + [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2}$$
$$\sigma_2 = \cosh\theta + a\sinh\theta - [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2} \tag{33a}$$

where

$$a = \tfrac{1}{2}K\rho(1-z)^{1/2} \qquad \theta = K(T-\rho)(1-z)^{1/2}. \tag{33b}$$

Using equation (28) we now calculate G_1 and G_2 and obtain

$$G_1 = \begin{cases} \rho[\cosh\theta + (2a)^{-1}\sinh\theta] \\ \quad +\tau\{a\sinh\theta + [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2}\} \qquad (0 \leqslant \tau \leqslant \rho) \\ \rho[\cosh\theta + (2a)^{-1}\sinh\theta]\cosh\varphi + \rho\{a\sinh\theta + [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2}\} \\ \quad \times[\cosh\varphi + (2a)^{-1}\sinh\varphi] \qquad (\rho \leqslant \tau \leqslant T) \end{cases} \tag{34a}$$

where $\varphi = K(\tau-\rho)(1-z)^{1/2}$

$$G_2 = \begin{cases} \rho[\cosh\theta + (2a)^{-1}\sinh\theta] \\ \quad +\tau\{a\sinh\theta - [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2}\} \qquad (0 \leqslant \tau \leqslant \rho) \\ \rho[\cosh\theta + (2a)^{-1}\sinh\theta]\cosh\varphi + \rho\{a\sinh\theta - [(1+a^2)\sinh^2\theta + a\sinh 2\theta]^{1/2}\} \\ \quad \times[\cosh\varphi + (2a)^{-1}\sinh\varphi] \qquad (\rho \leqslant \tau \leqslant T). \end{cases} \tag{34b}$$

These forms can now be used in conjunction with equations (29) and (13) to calculate $F(\tau' z)$ for specified initial boundary conditions, or in conjunction with equation (30) to obtain the long-term behaviour of F.

Now, it is clear from the structure of equations (34) that the expressions for F will be (algebraically) rather complicated and we therefore proceed to look at a particular limiting form of equation (31) for $W(\tau)$. We note that the total particle input into the system over a period T is proportional to $K^2(T-\rho)$, and we therefore consider the situation when $K \to \infty$ and $T - \rho \to 0$ with

$$J = K^2(T-\rho) \tag{35}$$

remaining finite. This means that there is a very high rate of particle injection into the system over a very short interval of time at the end of each cycle with the total particle input per cycle being specified. Experimentally this corresponds to intense short bursts of particle input, each of length $\mu(= T-\rho)$ injected at regular intervals T. The results we obtain will then apply whatever profile the input has over the interval μ as long as (a) $\mu \ll T$ and (b) μ is sufficiently small for there to be no effective coagulation of the particles during a single burst. As a consequence of this limiting procedure it is clear from equations (33*b*) that $\theta \to 0$ (with $\cosh\theta \to 1$, and $\sinh\theta \to \theta$), $a \to \infty$ and $a\theta \to L(1-z)$ with $L = (1/2)JT$. Equations (33*a*) then yield

$$\begin{aligned} \sigma_1 &= 1 + L(1-z) + [L^2(1-z)^2 + 2L(1-z)]^{1/2} \\ \sigma_2 &= 1 + L(1-z) - [L^2(1-z)^2 + 2L(1-z)]^{1/2} \end{aligned} \tag{36}$$

while equations (34) give for $0 \leqslant \tau < T$

$$\begin{aligned} G_1 &= T + \{L(1-z) + [L^2(1-z)^2 + 2L(1-z)]^{1/2}\}\tau \\ G_2 &= T + \{L(1-z) - [L^2(1-z)^2 + 2L(1-z)]^{1/2}\}\tau. \end{aligned} \tag{37}$$

For the remainder of our discussion we shall focus on the long-term behaviour of F which, from equation (30), is given by

$$F(\tau + nT, z) = \frac{(1-2\nu)L(1-z) + [L^2(1-z)^2 + 2L(1-z)]^{1/2}}{T[1 + 2\nu(1-\nu)L(1-z)]} \qquad (0 \leqslant \tau < T) \tag{38}$$

where $\nu = \tau/T$. We begin by considering the total particle number $N(\tau + nT)$ which from equation (8) is given by

$$M(\tau) = \frac{1}{2} t_0 P N(\tau + nT) = F(\tau + nT, 0) = \frac{(1 - 2\nu)L + (L^2 + 2L)^{1/2}}{T[1 + 2\nu(1 - \nu)L]}. \tag{39}$$

It is clear from this that as τ increases from 0 to T the value of $M(\tau)$ decreases from

$$[(L^2 + 2L)^{1/2} + L] \text{ to } [(L^2 + 2L)^{1/2} - L].$$

At this point it is instructive, as well as providing a useful check on our technique, to show how result (39) can be derived by an independent approach to the problem. The quantity $M(\tau)$ satisfies the usual coagulation equation $\mathrm{d}M/\mathrm{d}\tau = -M^2$ in the interval $0 \leqslant \tau < T$ and thus

$$M(\tau) = \frac{M(0)}{1 + M(0)\tau} \qquad (0 \leqslant \tau < T). \tag{40}$$

At $\tau = T$ the value of M is instantaneously increased by an amount J due to the 'delta function' particle source, and for the periodic solution we are currently investigating this will restore the value of M to $M(0)$. Hence

$$\frac{M(0)}{1 + M(0)T} + J = M(0). \tag{41}$$

This yields a quadratic equation for $M(0)$ whose positive root substituted into equation (40) gives the result (39).

As regards m_r, it follows from the discussion of section (2) that $m_r(\tau + nT)$ will be minus the coefficient of z^r in a power series expansion in z of the function $F(\tau + nT, z)$ given by equation (38). For general r the simplest expression thus obtained for m_r takes the form of a very unwieldly double summation, the details of which we will not give here. Rather, to illustrate the theory we quote now the expressions obtained for m_1 and m_2. These are

$$m_1 = \frac{1}{T[1 + 2\nu(1 - \nu)L]} \left\{ \frac{L^{1/2}(L + 1)}{(L + 2)^{1/2}} + \frac{(1 - 2\nu)L - 2\nu(1 - \nu)L^{3/2}(L + 2)^{1/2}}{1 + 2\nu(1 - \nu)L} \right\} \tag{42a}$$

$$m_2 = \frac{1}{T[1 + 2\nu(1 - \nu)L]} \left\{ \frac{L^{1/2}}{2(L + 2)^{3/2}} + \frac{2\nu(1 - \nu)L}{[1 + 2\nu(1 - \nu)L]^2} \times \left[(1 - 2\nu)L + \frac{L^{1/2}\{1 + [1 - 2\nu(1 - \nu)]L\}}{(L + 2)^{1/2}} \right] \right\} \tag{42b}$$

and beyond this point the expressions for m_r rapidly become more complicated with increasing r.

Finally, we consider the two limiting cases of $L \ll 1$ and $L \gg 1$. For $L \ll 1$ there is only a very small amount of coagulation within the period T and we would therefore expect that the results for m_r and M would agree with the solution of the standard steady-state coagulation equation with constant source term J/T. It is readily shown from equations (38) and (39) that this is so. For $L \gg 1$, on the other hand, the large amount of coagulation within the period T means that the number of particles at the end of the period, just before the injection of the next burst, is effectively zero, and thus within the interval $0 < t < T$ we would expect m_r and M to agree with the solution of the standard source-free time-dependent coagulation equation with initial particle number J. Again, from equations (38) and (39) this is readily shown to be true.

6. Discussion

In this paper we have developed the solution of the nonlinear coagulation equation in the presence of a source term, by reducing it to a linear equation, albeit of higher order. This has allowed us to consider (in section 3) situations where the long-term behaviour of the solution depends only on the source term and is independent of the initial boundary conditions, while in section 4 we have shown that for a periodic source term the long-term behaviour of the solution is also periodic. The technique has also allowed us to obtain analytic solutions of the problem for some simple source terms—a worthwhile task in view of the efforts that workers in the field have put into deriving analytic solutions of the coagulation equation in the absence of a source term (see Drake 1972, Williams and Loyalka 1991).

Let us now consider briefly the situation where the nature of the source term precludes a simple analytic approach and for which a numerical technique is therefore indicated. We have discussed in section 3 the existence of long-term solutions independent of the initial conditions and a detailed numerical investigation of such solutions is clearly of prime importance. One possible line of approach to this would be to tackle numerically the original coagulation equation (1) or equation (7) for F. This would however necessitate specifying sets of particular initial conditions and endeavouring to extract from the results a long-term solution independent of these conditions. A preferable approach might be to deal with equation (11). If solutions of this, G_1 and G_2 can be found satisfying the condition $\lim_{t\to\infty}(G_2/G_1) = 0 = \lim_{t\to\infty}(G_2'/G_1')$, then the long-term form for F is given immediately by G_1'/G_1, as discussed in section 3. Such an approach would certainly appear to be more direct when the form of the source term leads to equation (11) having solutions which are standard functions with known properties. For the case of a periodic source term we can improve on this approach of extrapolating numerically the long-term behaviour of $F(\tau', z)$ by using the result (29). For values of n for which the second terms in the numerator and the denominator are small compared with the respective first terms, knowledge of $F(\tau', z)$ for given τ and three consecutive integer values of n readily allows elimination of these second terms, so that the long-term behaviour of F as given by equation (30) can easily be obtained. Here, again, if the time variation of the source term is sufficiently simple it may be possible to use equation (30) with G_1 being a standard function. Thus, for example, if $W(\tau)$ involves a cosine variation in τ, then G_1 will involve Mathieu functions which can be used in evaluating equation (30).

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